MULTIVARIATE LIFE DISTRIBUTIONS INDUCED BY SHOT-NOISE PROCESS ENVIRONMENTS

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Abstract

We develop a class of univariate and multivariate life distributions for components and systems which operate in an unknown environment. The environment is assumed to be dynamic and is described by a shot-noise process. Special cases result in a univariate distribution with monotone hazard rates, and what appears to be a new family of multivariate exponential distributions with a singular component.

Key words and phrases: Dependence, Reliability, Survival, Multivariate Distributions, Exponential Distribution, Damage Accumulation.

1. Introduction

Consider a single or a two-component system which operates in an environment that may or may not be identical to the test bench environment. The component lifelengths, when assessed under the test bench environment, are assumed to have failure rates $\lambda_i(u) > 0$, $i=1, 2$, for $u \geq 0$. We shall assume that the nett effect of the operating environment is to modulate $\lambda_i(u)$ to $\lambda_i(u)\,\eta(u)$, $i=1, 2$, where $\eta(u)$ is unknown for all $u \geq 0$. If at any time u, the operating environment is judged harsher (gentler) than the test bench environment, then $\eta(u) > (<) 1$; $\eta(u) = 1$ would correspond to the case in which the operating environment and the test bench environment are identical. Given $\lambda_1(u)$, $\lambda_2(u)$ and $\eta(u)$, for all $u \geq 0$, we shall judge the component lifelengths independent. When $\eta(u)$, $u \geq 0$, is unknown, we shall describe our uncertainty about it by a suitable stochastic process $\{\eta(u), u \geq 0\}$. Under such circumstances, there is an induced dependence between the lifelengths of the two components, the nature of which depends on our choice of $\{\eta(u), u \geq 0\}$. The case $\lambda_i(u) = \lambda_i$, $i=1, 2$, with $\eta(u) = \eta$, and uncertainty about η described by a gamma distribution, has been considered by Lindley and Singpurwalla (1986), Currit and Singpurwalla (1989) and the references therein; the case wherein $\{\eta(u), u \geq 0\}$ is described by a gamma process has been considered by Singpurwalla and Youngren (1989). In this paper, we shall suppose that $\{\eta(u), u \geq 0\}$ is described by a "shot-noise process", a motivation for which is given below.

Suppose that the operating environment consists of a series of events, called "shots" or "jolts", whose effect is to induce stresses of unknown magnitudes X_k, $k=0, 1, \ldots$, on the components. The jolts occur over time according to a Poisson process with a known rate $m(u)$, $u \geq 0$, and suppose that whenever a stress of magnitude X occurs at an epoch s, then its contribution to $\eta(s+u)$ is $Xh(s+u)$, where the *attenuation function* $h(\cdot)$ is positive and nonincreasing. Thus, if $T_{(0)} < T_{(1)} < T_{(2)} < \ldots$, are the successive epochs at which stresses of magnitudes $X_0, X_1, X_2, \ldots$, respectively, occur, then

$$\eta(u) = \sum_{k=0}^{\infty} X_k \, h(u - T_{(k)}),$$

with $h(u) = 0$, for $u < 0$. The process $\{\eta(u), u \geq 0\}$ is called a ***shot-noise process***; see Cox and Isham (1980), p. 135.

As an example of the above, it may happen that an unexpected surge of power in a control system will temporarily increase the likelihood of failure, but the overload often decays, lowering the likelihood of failure. Another example pertains to the mortality rate of individuals suffering a heart attack. Since the heart muscle repairs itself, the rate declines with elapsed time since trauma, but the successive bouts of attack have a cumulative effect on the rate. Situations which necessitate a consideration of "cumulative damage" as the cause of failure can be incorporated within the framework of shot-noise processes by making $h(u)$ a constant, for all $u \geq 0$.

When $\lambda_i(u)$, the *inherent (or base-line) failure rate* is such that $\lambda_i(u) = \lambda_i$, for $u \geq 0$, then assuming that $\{\eta(u), u \geq 0\}$ is a shot-noise process is equivalent to assuming that $\lambda_i \eta(u)$, the failure rate of the i-th component under the operating environment, is also a shot-noise process. To ensure that the failure rate is not annihilated during the initial phase of the shot-noise process, we require that $T_{(0)} \equiv 0$.

We also need to make the following assumptions:

A1. $X_k \perp\!\!\!\perp T_{(k)}$, for all k, where "$\perp\!\!\!\perp$" denotes independence;

A2. $X_i \perp\!\!\!\perp X_j$, for all $i \neq j$, with X_i's having a common distribution G, and

A3. $m(t) > 0$ and is almost everywhere differentiable.

2. The Survival Functions

2.1 The Univariate Case

Let L denote the lifelength of a single component system whose inherent failure rate is λ, and which operates in an environment for which $\{\eta(u), u \geq 0\}$ is a shot-noise process with the parameters described in Section 1; assume that A1 - A3 hold. Let $M(t) = \int_0^t m(u)du$, $H(t) = \int_0^t h(u)du$, and suppose that G^* is the Laplace transform of G; then,

Theorem 2.1. Given λ, $M(t)$ and $H(t)$, $\overline{F}(\tau | \lambda, M(t), H(t))$, the survival function of L at $\tau \geq 0$, is

$$G^*\{\lambda H(\tau)\} \bullet \exp\left[-M(\tau) + \int_0^\tau G^*\{\lambda H(\tau)\}\, m(\tau - u)du\right].$$

Lemoine and Wenocur (1986) outline a proof of the above theorem, but have overlooked the requirement that $T_{(0)}$ must be 0. The proof is facilitated by

Lemma 2.2. Consider the time interval $[0, \tau)$, and let $T_{(1)} < T_{(2)} < \ldots < T_{(n)}$ be the epochs of events in a nonhomogeneous Poisson process with rate $m(u)$, $u \geq 0$. Then, if M^{-1} exists, T_i, $i = 1, \ldots, n$ the unordered epochs, are independent and identically distributed as a random variable U whose density at u, $o \leq u < \tau$, is $m(u)/M(\tau)$.

Proof of Lemma 2.2. If we rescale the time axis by M(t), then, on the new time scale, the epochs $M(T_{(1)}) < M(T_{(2)}) < \ldots$, are Poisson with rate 1 [see Cox and Isham (1980), p. 48]. Thus, given n, $M(T_{(1)}) < M(T_{(2)}) < \cdots < M(T_{(n)})$ are distributed as the order statistics in a sample of size n from a uniform distribution on $[0, M(\tau))$, from which it follows that the unordered epochs $M(T_i)$, $i=1, \ldots, n$, are independent and uniform on $[0, M(\tau))$. Hence, given that M^{-1} exists, $P(M(T_i) \le u) = P(T_i \le M^{-1}(u)) = u|M(\tau) = M\ M^{-1}(u)|M(\tau)$, and so $P(T_i \le u) = M(u)|M(\tau)$, from which the result follows. ||

Proof of Theorem 2.1. Suppose that n jolts occur over the time interval $[0, \tau)$ at epochs $\underset{\sim}{T} = (T_{(0)} < T_{(1)} < \cdots < T_{(n)}$, with each jolt inducing a stress of magnitude X. Given λ, n, $\underset{\sim}{T}$ and X, the survival function of L at time τ, is

$$\exp\left[-\left\{\lambda\ X\ H(\tau) + \lambda \sum_{k=1}^{n} X\ H(\tau - T_{(k)})\right\}\right],$$

which because of A2 and Lemma 2.2 can also be written as

$$\exp[-\ \{\lambda\ X\ H(\tau)\}]\ [\exp\{-\ \lambda\ X\ H(\tau\ -\ u)\}]^{n}.$$

Unconditioning on X, and then on U, the above becomes

$$G^{*}(\lambda\ H(\tau)) \left[\int_0^{\tau} G^{*}\big(\lambda\ H(\tau - u)\big)\ \frac{m(u)}{M(\tau)}\ du\right]^{n}.$$

Our final step is to uncondition on n – which has the probability mass function $e^{-M(\tau)}\ (M(\tau))^{n}/n!$ – and to simplify the ensuing expressions. ||

2.2 The Bivariate Case

Let L_i, $i=1, 2$, denote the lifelengths of a two component system, with the i-th component having λ_i as its inherent failure rate. Suppose that the system operates in the environment described before. Then,

Theorem 2.3. Given λ_1, λ_2, M(t) and H(t), $\overline{F}(\tau_1, \tau_2 | \lambda_1, \lambda_2, M(t), H(t))$, the joint survival function of L_1 and L_2, for $0 \le \tau_1 \le \tau_2$, is

$$G^*[\lambda_1 H(\tau_1) + \lambda_2 H(\tau_2)] \bullet \exp\left[\int_0^{\tau_1} G^*\left[\lambda_1 H(\tau_1 - u_1) + \lambda_2 H(\tau_2 - u_1)\right] m(u_1) du_1\right]$$

$$\bullet \exp\left[\int_{\tau_1}^{\tau_2} G^*[\lambda_2 H(\tau_2 - u_2)] m(u_2) du_2 - M(\tau_2)\right].$$

The proof of this theorem is facilitated by an elaboration of Lemma 2.2.

Lemma 2.4. Consider the time interval (τ_1, τ_2), and let $T_{(n_1+1)} < T_{(n_1+2)} < \cdots < T_{(n_2)}$ be the epochs of events in a nonhomogeneous Poisson process with rate m(u), $u \ge 0$. Then, if M^{-1} exists, T_i, $i=(n_1+1), \ldots, (n_2)$, the unordered epochs are independent and identically distributed as a random variable U whose density at u, $\tau_1 \le u < \tau_2$ is $m(u)/(M(\tau_2) - M(\tau_1))$.

Proof of Lemma 2.4. As in Lemma 2.2, we rescale the time axis by M(t) and observe that on the new scale, given $(n_2 - n_1)$, the epochs $M(T_{(n_1+1)}) < M(T_{(n_1+2)}) < \cdots < M(T_{(n_2)})$ are distributed as the order statistics in a sample of size $(n_2 - n_1)$ from a uniform distribution on $[M(\tau_1), M(\tau_2))$; the proof now follows via arguments analogous to those in the proof of Lemma 2.2. ||

Proof of Theorem 2.3. Parallels that of Theorem 2.1, with the additional proviso that for a Poisson process, the number of events in $[\tau_1, \tau_2)$ is independent of the number of events in $[0, \tau_1)$.

Remark 2.5. The marginal survival functions of L_1 and L_2, which follow from Theorem 2.3 by setting τ_2 and τ_1 equal to zero, respectively, agree with the result of Theorem 2.1.

Remark 2.6. A generalization of Theorem 2.3 to the multivariate (multi-component) case is straightforward, with the $k > 2$ variate version calling for (k-2) additional parameters.

3. A New Bivariate Distribution with Exponential Marginals

A consideration of some special cases of Theorem 2.3 results in some interesting distributions, one of which is highlighted here. Suppose that for all $u \geq 0$, $h(u) \equiv 1$, $m(u) = m$, and G is a gamma distribution with shape α and scale b. Then, from Theorem 2.3, we see that for $\alpha = 1$, the joint survival function of L_1 and L_2, for $0 \leq \tau_1 \leq \tau_2$ is

$$\overline{F}(\tau_1, \tau_2 | \lambda_1, \lambda_2, b, m) = \left(\frac{b}{b+\lambda_1\tau_1+\lambda_2\tau_2}\right) \left(\frac{b+\lambda_2(\tau_2-\tau_1)}{b+\lambda_1\tau_1+\lambda_2\tau_2}\right)^{-mb/(\lambda_1+\lambda_2)} \bullet \left(\frac{b}{b+\lambda_2(\tau_2-\tau_1)}\right)^{-mb/\lambda_2} e^{-m\tau_2}, \tag{3.1}$$

and so the marginal survival function of L_2, for $\tau_2 \geq 0$, is

$$\overline{F}_2(\tau_2 | \lambda_2, b, m) = \left(\frac{b+\lambda_2\tau_2}{b}\right)^{\frac{mb}{\lambda_2} - 1} e^{-m\tau_2} . \tag{3.2}$$

Verify, from Theorem 3.1, that when $\alpha \neq 1$, the marginal survival function of L_2, for $\tau_2 \geq 0$, is of the form

$$\left(\frac{b}{b+\lambda\tau}\right)^{\alpha} \exp\left[\frac{mb}{\lambda(\alpha-1)}\left\{1 - \left(\frac{b}{b+\lambda\tau}\right)^{\alpha-1}\right\} - m\tau\right]. \tag{3.3}$$

If we assume that $\lambda_1 = \lambda_2 = \lambda$, and that $m = \lambda/b$, then for $0 \leq \tau_1 \leq \tau_2$, (3.1) becomes

$$\overline{F}(\tau_1, \tau_2|\lambda, b, m) = \sqrt{\frac{1 - m\tau_1 + m\tau_2}{1 + m\tau_1 + m\tau_2}}\; e^{-m\tau_2}, \tag{3.4}$$

and, for $\tau_2 \geq 0$, (3.2) becomes

$$\overline{F}_2(\tau_2|\lambda, b, m) = e^{-m\tau_2}, \tag{3.5}$$

an exponential distribution. The results for $\tau_2 \leq \tau_1$ and $\overline{F}_1(\tau_1|\lambda, b, m)$, the survival function of L_1, are symmetric, and so the joint survival function may be written

$$\overline{F}(\tau_1, \tau_2|\lambda, b, m) = \sqrt{\frac{1 - m\min(\tau_1,\tau_2) + m\max(\tau_1,\tau_2)}{1 + m(\tau_1 + \tau_2)}}\; e^{-m\max(\tau_1,\tau_2)}. \tag{3.6}$$

Thus, the nonageing property of a component in the test environment will be preserved in a shot-noise environment, if the stress inducing jolts are Poisson with a rate λ/b, if the stresses induce a cumulative damage on the component, and if the magnitudes of stress are exponential with a scale parameter b.

In Figure 3.1, we describe the behavior of $r_2(\tau_2)$, the failure rate function of (3.2) as a function of the relationship between λ_2/b and m. Verify that $r_2(\tau_2) = (\lambda_2/b)\,(1+m\tau)/(1+(\lambda_2/b)\tau)$, so that $r_2(\tau_2) \rightarrow m$ as $\tau_2 \rightarrow \infty$, and that $r_2(\tau_2)$ is increasing (decreasing) in τ_2 and asymptoting to m depending on whether $(\lambda_2/b) < (>)\ m$.

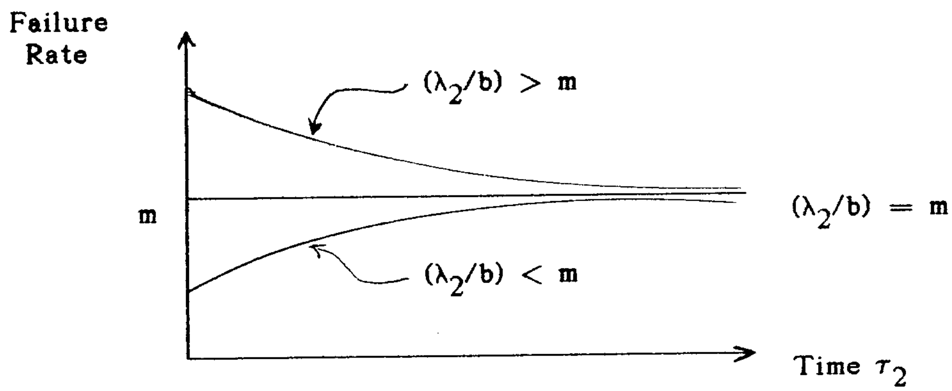

Figure 2.1. Behavior of the Failure Rate Function of $\overline{F}_2(\bullet)$

An intuitive explanation of the above behavior of the failure rate function, as a function of the relationship between (λ/b) and m is given in Youngren (1988). Of particular interest is the reason for its asymptoting to m. Our explanation here is that under a shot-noise process with cumulative damage, the failure of a component occurs when the damage exceeds a threshold. With the inter-arrival times between the jolts being exponential, the situation here is analogous to the behavior of a multicomponent system, with component lifelengths being exponential, for which the life distribution is a gamma, whose failure rate asymptotes to a constant.

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